

Aspects of superfluid cold atomic gases in optical lattices

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We review our studies on Bose and Fermi superfluids of cold atomic gases in optical lattices at zero temperature. Especially, we focus on superfluid Fermi gases along the crossover between the Bardeen-Cooper-Schrieffer (BCS) and the Bose-Einstein condensate (BEC) states, which enables us to study the Bose and Fermi superfluids in a unified point of view. We discuss basic static and long-wavelength properties (such as equation of state, incompressibility, and effective mass), energetic stability, and energy band structures of the superfluid Fermi gases in an optical lattice periodic along one spatial direction. The periodic potential makes pairs of atoms to be strongly bound, and this can affect the static and long-wavelength properties and the stability of the superflow. Regarding the band structure, a peculiar loop structure called “swallowtail” can appear in superfluid Fermi gases as well as in the Bose case while its mechanism of the emergence is very different from the bosonic counterpart. Other quantum phases that the cold atomic gases in optical lattices can show are also briefly discussed from their role as quantum simulators of the Hubbard models.

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I. INTRODUCTION

In 1995, the atomic physics witnessed the coolest state of matter realized in the laboratories to that day. With lasers and electromagnetic fields, atomic Bose gases were trapped and cooled down to within a micro-Kelvin to make them coalesce into a single quantum wave, known as a Bose-Einstein condensate (BEC) state [1]. With the technology and tools developed and upgraded, finally about 10 years later, physicists succeeded in cooling atomic Fermi gases down to the even lower temperatures and in providing a conclusive evidence that the interacting atomic Fermi gas could be controlled to be in the state from Bardeen-Cooper-Schrieffer (BCS) superfluid of loosely bound pairs to a BEC of tightly bound dimers [2]. This quantum wonderland of cold atomic gases has been proven to be rich in physics and it has also shown us new realms of research [3–7]. Unprecedented high-precision control over the cold atomic gases has made it possible to mimic various quantum systems. It has also paved a way to new physical parameter regimes which had not been attainable in other quantum systems and therefore had not been well thought of.

In the cold Fermi gas with an equal mixture of two hyperfine states (for simplicity, called as spin up/down), wide-range control of the short-range interatomic interaction (characterized by an s -wave scattering length a_s) via Feshbach resonances [8], has enabled mapping a new landscape of superfluidity, known as “BCS-BEC crossover”, smoothly connecting from the BCS superfluid to the BEC superfluid [9]. For weakly attractive interaction ($1/k_F a_s \ll -1$, with k_F conventionally being the Fermi momentum of the free Fermi gas of the same density), the gas shows the superfluid behavior originating from the many-body physics of Cooper pairs which

are formed from weak pairing of atoms of opposite spin and momentum and whose spatial extent is larger than the interparticle distance. For strongly attractive interaction ($1/k_F a_s \gg 1$), the gas shows the superfluid behavior which can be explained by the Bose-Einstein condensation of the tightly-bound bosonic dimers made up of two fermions of opposite spin. In the crossover regime/unitary regime ($1/k_F |a_s| \lesssim 1$) bridging two well-understood regimes above, the gas is strongly interacting, which means that the interaction energy is comparable to the Fermi energy of the free Fermi gas of the same density and defies the perturbative many-body techniques. Especially, the physics in the so-called “unitary limit” ($|a_s| = \infty$) is universal because only one remaining length scale, $1/k_F$, which is approximately the interparticle distance, appears in the equation of state of the gas, while the interaction effect appears as a universal dimensionless parameter. Because the physics at unitarity is universal, i.e., independent of its constituents and is non-perturbative due to the strong interaction, the unitary Fermi gas has been the test-bed for the various theoretical techniques developed up to now (see, e.g., Sec. VB and VC in Ref. [6]).

Cold atomic gases also provide insights into other forms of matter. Inside neutron stars, there might be quark matter made of different ratios of quarks and this imbalance might result in new types of superfluid, such as Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phase which has spatially non-uniform order parameter [10]. Even though the electric-charge neutrality of atoms might change the picture, we might get some ideas from the investigation on imbalanced atomic Fermi gases [11]. Another example is the quark-gluon plasma (QGP) created in the Relativistic Heavy Ion Collider (RHIC). QGP of temperature about 10^{12} K was produced by smashing

nuclei together. Measurements on its expansion after its creation show that QGP is a nearly perfect fluid with very small shear viscosity. A cloud of atomic Fermi gases at unitarity shows the same strange behavior while the origins of the similarity between the hot and the cold matter are to be speculated [12].

Cold atoms in periodic potentials are more intriguing because of their possible connection to the solid state/condensed matter physics (see, e.g., Refs. [5, 13] for reviews). The artificial periodic potential of light, so-called “optical lattice”, is configured with pairs of counter-propagating laser fields which are detuned from atomic transition frequencies large enough that they act as practically free-of-defect and conservative potentials that the atoms experience via optical dipole force. Geometry and depth of the lattice potential can be controlled by orienting the directions and changing the intensities of the laser beams, respectively. Moreover, macroscopically large lattice constant of the optical lattice compared with the lattice potential in solids is a great advantage for experimental observation and manipulation; nowadays, *in situ* imaging and addressing of the strongly-correlated systems at the single-site level have been realized (e.g., Refs. [14]). The high controllability of both the optical lattice properties via laser fields and the interatomic interaction via the Feshbach resonance makes cold atomic gases serve as quantum simulators for various models of condensed matter physics. Many phenomena of solid state/condensed matter physics have already been observed or realized in cold atomic gases on the optical lattices, including band structure, Bloch oscillation, Landau-Zener tunneling, and superfluid-Mott insulator transition [15, 16]. The precision control and measurement at the level of one lattice site enable the cold atomic gases on optical lattice to be applied in inventing new manipulation techniques and novel devices such as matter-wave interferometry, optical lattice clocks, and quantum registers [17, 18].

In this review article, we mainly explore the superfluid properties of cold atomic gases in the optical lattice at zero temperature. Superfluidity is the most well-known quantum phase of ultracold atomic gases and is prevalent in many other systems such as superconductors, superfluid helium, and superfluid neutrons in “pasta” phases (see, e.g., Ref. [19]) of neutron star crusts. Knowing its properties is also important both for judging whether the superfluid state is approached in experiments and applying its properties in controlling the system. Depending on the physical conditions, cold atomic gases in optical lattices show various interesting phases other than the superfluid phase, for example, simulating Hubbard models and some other quantum phases will be explained very briefly in Sec. VI. In our discussion, we mainly consider atomic Fermi gases partly because weakly interacting atomic Bose gases can be considered as an extreme limit in the BCS-BEC crossover of the atomic Fermi gases and partly because physics of the cold Fermi gases is richer. Bose gases are discussed separately in the cases where we

cannot find the corresponding phenomena in the Fermi gases. This does not mean that the cold Bose gases are less interesting and less important. The tools and technique developed for creating the BEC were the building block for those for creating superfluid Fermi gases and superfluid properties of the Bose gases might be more useful in some applications due to its higher superfluid transition temperature.

In Sec. II, our system and the theoretical frameworks are presented. There, we explain the mean-field theory and the hydrodynamic theory depending on the validity regime of each theory. Using cold atomic gases, various parameter regimes become accessible and the mean-field theory in the continuum model is a powerful tool which allows us to study superfluid states covering such a wide region. Hydrodynamic theory, though its validity regime is limited, can provide precise analytic predictions in some limit, which can be served as complementary to the mean-field theory. Based on these methods, we study various aspects of the superfluid state in a lattice potential periodic along one spatial direction. This setup is the simplest but contains essential effects of the lattice potential.

Section III discusses basic static and long-wavelength properties of the cold Fermi gases in the optical lattice. Specifically, the incompressibility and the effective mass are obtained from the equation of state (EOS). Focusing on the unitary Fermi gas as a typical example, here we show that these properties in the lattice can be strongly modified from those in free space [29]. Formation of bound pairs is assisted by the periodic potential and this results in a qualitative change of the EOS.

In Sec. IV, we examine the stability of the superfluid flow in the optical lattice. Mainly the energetic instability and the corresponding critical velocity of the superflow are discussed. Stability of the superflow is a fundamental problem of the superfluidity and cold atomic gases allow us to study this important problem for various parameter regimes and for both the Bose and Fermi superfluids. Unlike an obstacle in uniform superfluids, periodic potential can modify the excitation spectrum and this manifests itself in the behavior of the critical velocity [27, 55].

In Sec. V, we investigate the energy band structures of cold atomic Fermi gases in an optical lattice along the BCS-BEC crossover. By tuning the interatomic interaction strength, the nonlinear effect of the interaction energy can be made to dominate over external periodic potential. In this parameter regime, “swallowtail” energy band structures emerge and the physical properties of the system is affected by their appearance [74]. This is one example that cold atomic gases can show physical phenomena inaccessible in other systems.

In Sec. VI, we discuss quantum phases of cold atomic gases in optical lattices and show how they play their role as “quantum simulators” of Hubbard models. Before the advent of the cold atomic gases in optical lattices, the Hubbard model has been considered as an approximate

toy model of more complicated real system and been one of the central research topics in solid-state/condensed-matter physics. High-precision control of the cold atomic gases in optical lattices now makes the Hubbard models realized in experiments and we might address open questions of quantum magnetism and high T_c superconductivity.

Finally, Sec. VII contains the summary and some perspectives on the cold atomic gases in optical lattices.

II. THEORETICAL FRAMEWORKS

A. Setup of the system and periodic potential

In the present article, we discuss superfluid flow made of either fermionic or bosonic cold atomic gases subject to an optical lattice. For concreteness, we consider one of the most typical cases: the system is three dimensional (3D) and the potential is periodic in one dimension with the following form:

$$V_{\text{ext}}(\mathbf{r}) = sE_R \sin^2 q_B z \equiv V_0 \sin^2 q_B z. \quad (1)$$

Here, $V_0 \equiv sE_R$ is the lattice height, s is the lattice intensity in dimensionless units, $E_R = \hbar^2 q_B^2 / 2m$ is the recoil energy, m is the mass of atoms, $q_B = \pi/d$ is the Bragg wave vector, and d is the lattice constant. For simplicity, we also assume that the supercurrent is in the z direction and thus the system is uniform in the transverse (i.e., x and y) directions. Throughout the present article, we set the temperature $T = 0$.

Before giving a detailed description of the theoretical framework, it is useful to summarize the scales in this system. The periodic potential is characterized by two energy scales: one is the recoil energy E_R , which is directly related to the lattice constant d , and the other is the lattice height $V_0 = sE_R$. Regarding Bose-Einstein condensates of bosonic atoms, a characteristic energy scale is the interaction energy gn , where $g = 4\pi\hbar^2 a_s / m$ is the interaction strength and n is the density of atoms. On the other hand, in the case of superfluid Fermi gases, the total energy is on the order of the Fermi energy $E_F = \hbar^2 k_F^2 / (2m)$ with the Fermi wave number $k_F \equiv (3\pi^2 n)^{1/3}$ corresponding to the uniform non-interacting Fermi gas with the same density n . Therefore, the relative effect of the lattice strength is given by the ratio $\eta_{\text{height}} = V_0 / gn$ for bosons and $\eta_{\text{height}} = V_0 / E_F$ for fermions. Likewise, relative fineness of the lattice (compared to the healing length) is characterized by the ratio $\eta_{\text{fine}} = (gn / E_R)^{-1} \sim (\xi / d)^2$ for bosons and, for fermions, $\eta_{\text{fine}} = (E_F / E_R)^{-1} \sim (k_F d)^{-2}$, which is at most $\sim (\xi / d)^2$ near unitarity. Here, ξ is the healing length given as $\xi = \hbar / (2mg n)^{1/2}$ for Bose superfluids and as $\xi \sim k_F^{-1}$ for Fermi superfluids at unitarity, which is consistent with the BCS coherence length $\xi_{\text{BCS}} = \hbar v_F / \Delta$, where $v_F = \hbar k_F / m$ and Δ is the pairing gap. We can also say that the validity of the local density approximation

(LDA) is characterized by $1/\eta_{\text{fine}} \gtrsim (d/\xi)^2 \gg 1$ corresponding to the lattice with low fineness $\eta_{\text{fine}} \ll 1$.

In the present article, we shall consider a large parameter region covering weak to strong lattices, $\eta_{\text{height}} = O(10^{-2}) - O(10)$ ($s \sim 0.1 - 5$), and low to high fine lattices, $\eta_{\text{fine}} \sim 0.1 - 10$.

B. Mean-field theory in the continuum model

To study the superfluidity in such various regimes in a unified manner, one of the most useful theoretical frameworks is the mean-field theory in the continuum model. Since our system is a superfluid in three dimensions and the number of particles in each site is infinite in our setup, effects of quantum fluctuations, which are not captured by the mean-field theory, may be small. It is also noted that widely-used tight-binding models are invalid in the weak lattice regime of $\eta_{\text{height}} \lesssim 1$.

For dilute BECs at zero temperature, the system is well described by the Gross-Pitaevskii (GP) equation [4, 20, 21]:

$$-\frac{\hbar^2}{2m} \partial_z^2 \Psi + V_{\text{ext}}(z) \Psi + g |\Psi|^2 \Psi = \mu \Psi, \quad (2)$$

where $\Psi(z) = \sqrt{n(z)} \exp[i\phi(z)]$ is the condensate wave function, $\phi(z)$ is its phase, and μ is the chemical potential. The local superfluid velocity is given by $v(z) = (\hbar/m) \partial_z \phi(z)$.

For superfluid Fermi gases, we consider a balanced system of attractively interacting (pseudo)spin 1/2 fermions, where the density of each spin component is $n/2$. To describe the BCS-BEC crossover at zero temperature, we use the Bogoliubov-de Gennes (BdG) equations [6, 22],

$$\begin{pmatrix} H'(\mathbf{r}) & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H'(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \epsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}, \quad (3)$$

with $H'(\mathbf{r}) = -\hbar^2 \nabla^2 / 2m + V_{\text{ext}} - \mu$, $u_i(\mathbf{r})$ and $v_i(\mathbf{r})$ are quasiparticle wave functions, and ϵ_i is the corresponding quasiparticle energy. The pairing field $\Delta(\mathbf{r})$ and the chemical potential μ in Eq. (3) are self-consistently determined from the gap equation

$$\Delta(\mathbf{r}) = -g \sum_i u_i(\mathbf{r}) v_i^*(\mathbf{r}), \quad (4)$$

together with the constraint on the average number density

$$\bar{n} = \frac{2}{V} \sum_i \int |v_i(\mathbf{r})|^2 d\mathbf{r} = \frac{1}{V} \int n(\mathbf{r}) d\mathbf{r}, \quad (5)$$

with $n(\mathbf{r}) \equiv 2 \sum_i |v_i(\mathbf{r})|^2$. Here g is the coupling constant for the contact interaction and V is the volume of the system. The average energy density \bar{e} can be calculated

as

$$\bar{e} = \frac{1}{V} \int d\mathbf{r} \left[\frac{\hbar^2}{2m} \left(2 \sum_i |\nabla v_i|^2 \right) + V_{\text{ext}} n(\mathbf{r}) + \frac{1}{g} |\Delta(\mathbf{r})|^2 \right]. \quad (6)$$

For contact interactions, the right hand side of Eq. (4) has ultraviolet divergence, which has to be regularized by replacing the bare coupling constant g with the two-body T -matrix related to the s -wave scattering length [23–26]. A standard scheme [23] is to introduce the cutoff energy $E_C \equiv \hbar^2 k_C^2 / 2m$ in the sum over the BdG eigenstates and replace g by the following relation:

$$\frac{1}{g} = \frac{m}{4\pi\hbar^2 a_s} - \sum_{k < k_C} \frac{1}{2\epsilon_k^{(0)}}, \quad (7)$$

with $\epsilon_k^{(0)} \equiv \hbar^2 k^2 / 2m$.

In the presence of a supercurrent with wavevector $Q = P/\hbar$ (P is the quasimomentum for atoms rather than for pairs, thus it is defined in the range of $|P| \leq \hbar q_B / 2$) in the z direction, one can write the quasiparticle wavefunctions in the Bloch form as $u_i(\mathbf{r}) = \tilde{u}_i(z) e^{iQz} e^{i\mathbf{k}\cdot\mathbf{r}}$ and $v_i(\mathbf{r}) = \tilde{v}_i(z) e^{-iQz} e^{i\mathbf{k}\cdot\mathbf{r}}$ leading to the pairing field as

$$\Delta(\mathbf{r}) = e^{i2Qz} \tilde{\Delta}(z). \quad (8)$$

Here $\tilde{\Delta}(z)$, $\tilde{u}_i(z)$, and $\tilde{v}_i(z)$ are complex functions with period d and the wave vector k_z ($|k_z| \leq q_B$) lies in the first Brillouin zone. This Bloch decomposition transforms Eq. (3) into the following BdG equations for $\tilde{u}_i(z)$ and $\tilde{v}_i(z)$:

$$\begin{pmatrix} \tilde{H}_Q(z) & \tilde{\Delta}(z) \\ \tilde{\Delta}^*(z) & -\tilde{H}_{-Q}(z) \end{pmatrix} \begin{pmatrix} \tilde{u}_i(z) \\ \tilde{v}_i(z) \end{pmatrix} = \epsilon_i \begin{pmatrix} \tilde{u}_i(z) \\ \tilde{v}_i(z) \end{pmatrix}, \quad (9)$$

where

$$\tilde{H}_Q(z) \equiv \frac{\hbar^2}{2m} \left[k_\perp^2 + (-i\partial_z + Q + k_z)^2 \right] + V_{\text{ext}}(z) - \mu.$$

Here, $k_\perp^2 \equiv k_x^2 + k_y^2$ and the label i represents the wave vector \mathbf{k} as well as the band index.

C. Hydrodynamic theory

When the local density approximation (LDA) is valid, such that the typical length scale of the density variation given by d is much larger than the healing length ξ of the superfluid, hydrodynamic theory in the LDA can be useful [27]. In hydrodynamic theory, we describe the system in terms of the density field $n(z)$ and the (quasi-)momentum field $P(z)$ [or the velocity field $v(z)$]. The LDA assumes that, locally, the system behaves like a uniform gas; thus the energy density $e(n, P)$ can be written in the form

$$e(n, P) = nP^2/2m + e(n, 0) \quad (10)$$

and one can define the local chemical potential $\mu(n) = \partial e(n, 0) / \partial n$. The density profile of the gas at rest in the presence of the external potential can be obtained from the Thomas-Fermi relation: $\mu_0 = \mu[n(z)] + V_{\text{ext}}(z)$. If the gas is flowing with a constant current density $j = n(z)v(z)$, the Bernoulli equation for the stationary velocity field $v(z)$ is

$$\mu_j = \frac{m}{2} \left[\frac{j}{n(z)} \right]^2 + \mu(n) + V_{\text{ext}}(z), \quad (11)$$

where μ_j is the z -independent value of the chemical potential.

In the following two typical cases, the uniform gas has a polytropic equation of state:

$$\mu(n) = \alpha n^\gamma. \quad (12)$$

1) A dilute Bose gas with repulsive interaction, where $\gamma = 1$ and $\alpha = g = 4\pi\hbar^2 a_s / m$. 2) A dilute Fermi gas at unitarity, where $\mu(n) = (1 + \beta)E_F = [(1 + \beta)(3\pi^2)^{2/3}\hbar^2/(2m)]n^{2/3}$, i.e., $\gamma = 2/3$ and $\alpha = (1 + \beta)(3\pi^2)^{2/3}\hbar^2/(2m)$. Here β is a universal parameter, which is negative and its absolute value is of order unity accounting for attractive interatomic interactions [6, 28].

Using the equation of state one can write

$$mc_s^2(z) = n \frac{\partial}{\partial n} \mu(n) = \gamma \mu(n), \quad (13)$$

where $c_s(z)$ is the local sound velocity, which depends on z through the density profile $n(z)$. In a uniform gas of density n , the sound velocity is given by $c_s^{(0)} = [\gamma \mu(n)/m]^{1/2}$.

III. EQUATION OF STATE, INCOMPRESSIBILITY, AND EFFECTIVE MASS

In this section, focusing on superfluid Fermi gases at unitarity, we discuss effects of the periodic potential on macroscopic and static properties of the fluid such as equation of state, incompressibility, and effective mass [29]. The important point is that the periodic potential favors the formation of bound molecules in a two-component Fermi gas even at unitarity [30] (see also, e.g., Refs. [31, 32]). The emergence of the lattice-induced bound states drastically changes the above macroscopic and static properties from those of uniform systems in the strong lattice regime of $\eta_{\text{height}} \gg 1$ [29]. Such an effect is absent in ideal Fermi gases and BECs of repulsively interacting bosonic atoms which can be considered as two limits in the BCS-BEC crossover.

A. Basic equations

At zero temperature, the chemical potential μ (the equation of state) and the current j of a superfluid Fermi

gas in a lattice are given by the derivative of the average energy density $\bar{e} = E/V$ with respect to the average (coarse-grained) density \bar{n} [33] and the average quasimomentum \bar{P} of the bulk superflow, respectively (hereafter, for notational simplicity, we omit “-” for the coarse-grained quantities, which should not be confused with the local quantities):

$$\mu = \frac{\partial e(n, P)}{\partial n}; \quad j = \frac{\partial e(n, P)}{\partial P}. \quad (14)$$

The incompressibility (or inverse compressibility) κ^{-1} and the effective mass m^* , are given by the second derivatives of e with respect to n and P :

$$\kappa^{-1} = n \frac{\partial^2 e(n, P)}{\partial n^2} = n \frac{\partial \mu(n, P)}{\partial n}, \quad (15)$$

$$\frac{1}{m^*} = \frac{1}{n} \frac{\partial^2 e(n, P)}{\partial P^2} = \frac{1}{n} \frac{\partial j(n, P)}{\partial P}. \quad (16)$$

We calculate these quantities for $P = 0$, i.e. for a gas at rest, in the periodic potential.

In the absence of the lattice potential ($s = 0$), thermodynamic properties of unitary Fermi gases show a universal behavior: the only relevant length scale is the interparticle distance fixed by k_F . Due to translational invariance, one can write $e(n, P) = e(n, 0) + nP^2/2m$ so that $j = nP/m$ and $m^* = m$. Furthermore, the energy density at $P = 0$ can be written as $e(n, 0) = (1+\beta)e^0(n, 0)$, where $e^0(n, 0) \equiv (3/5)nE_F \propto n^{5/3}$ is the energy density of the ideal Fermi gas. Thus we have $\mu = (1+\beta)E_F + P^2/2m$ and $\kappa^{-1} = (2/3)(1+\beta)E_F$.

B. Equation of state and density profile

When the lattice height s is large, the periodic potential favors the formation of bound molecules. In the strong lattice limit $\eta_{\text{height}} = sE_R/E_F \gg 1$, the system tends to be a BEC of lattice-induced bosonic molecules. Therefore, in this regime, the chemical potential shows a linear density dependence, $\mu \propto n$, as shown by the red solid line in the inset of Fig. 1 calculated for unitary Fermi gases in a lattice with $s = 5$. This is clearly different from the density dependence of the chemical potential in the uniform system ($s = 0$), $\mu \propto n^{2/3}$, as shown by the blue dashed line in the same inset. It is also noted that, for $s \gg 1$, this linear density dependence persists even at relatively large densities so that $E_F/sE_R \sim 1$ (e.g., $nq_B^{-3} = 0.1$ corresponds to $E_F/sE_R \simeq 2.1$ and thus $E_F/sE_R \simeq 0.41$ in the case of this figure) because the system effectively behaves like a 2D system in this density region due to the bandgap in the longitudinal degree of freedom.

This drastic change of EOS manifests itself as a change of the coarse-grained density profile when a harmonic confinement potential $V_{\text{ho}}(\mathbf{r})$ is added to the periodic potential. The coarse-grained density profile, $n(\mathbf{r})$, is calculated using the LDA for μ :

$$\mu_0 = \mu[n(\mathbf{r})] + V_{\text{ho}}(\mathbf{r}). \quad (17)$$

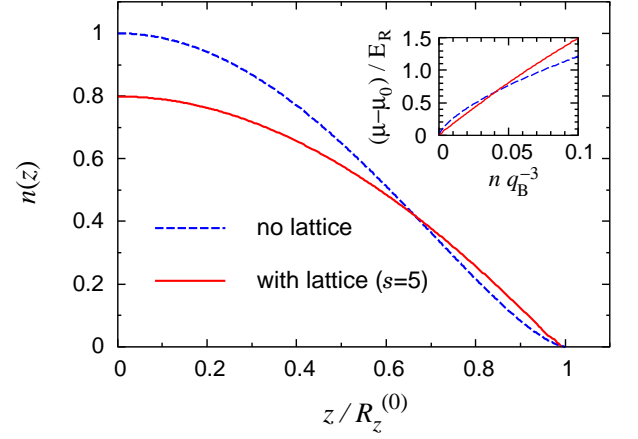


FIG. 1: (Color online) Coarse-grained density profiles of a trapped unitary Fermi gas, $n(r_{\perp} = 0, z)$ for $s = 0$ and 5 in units of the central density $n(0) = 0.0869q_B^3$ calculated for $s = 0$ (this local density corresponds to $E_F/E_R = 1.88$). The quantity $R_z^{(0)}$ is the axial Thomas-Fermi radius for $s = 0$. The inset shows the density dependence of the chemical potential of unitary Fermi gases. Here μ_0 is the chemical potential in the limit of $n = 0$. This figure is taken from Ref. [29]

Here $\mu[n]$ is the local chemical potential as a function of the coarse-grained density n obtained by the BdG calculation for the lattice system and μ_0 is the chemical potential of the system fixed by the normalization condition $\int d\mathbf{r} n(\mathbf{r}) = N$. Figure 1 clearly shows that, for $s = 5$, the profile takes the form of an inverted parabola, reflecting the linear density dependence of the chemical potential (see inset). In this calculation, we assume an isotropic harmonic potential, $V_{\text{ho}}(\mathbf{r}) = m\omega^2 r^2/2$, where ω is the trapping frequency, $\hbar\omega/E_R = 0.01$, and the number of particles $N = 10^6$; these parameters are close to the experimental ones in Ref. [34].

C. Incompressibility and effective mass

The formation of molecules induced by the lattice has important consequences also on κ^{-1} and m^* . Due to the linear density dependence of the chemical potential in the strong lattice regime ($\eta_{\text{height}} \gg 1$ or $E_F/E_R \ll s$; or low density limit for a fixed value of s), κ^{-1} is also proportional to n and $\kappa^{-1}/\kappa^{-1}(s=0) \propto n^{1/3} \rightarrow 0$ for $E_F/E_R \rightarrow 0$ [see Fig. 2(a)]. This means that the gas becomes highly compressible in the presence of a strong lattice. This is in a strong contrast to the ideal Fermi gas corresponding to the BCS limit, which gives nonzero values of $\kappa^{-1}/\kappa^{-1}(s=0) \sim 1$ even in the same limit (see Fig. 2 in Ref. [29]). On the other hand, in the weak lattice limit (or high density limit for a fixed value of s), the system reduces to a uniform gas. By using an hydrodynamic theory, which is valid when $E_F/E_R \gg 1$, and expanding with respect to the small parameter sE_R/E_F ,

we obtain κ^{-1} of unitary Fermi gases in this regime as [29]

$$\kappa^{-1} \simeq \frac{2}{3}(1+\beta)E_F \left[1 + \frac{1}{32}(1+\beta)^{-2} \left(\frac{sE_R}{E_F} \right)^2 \right] + O[(sE_R/E_F)^4]. \quad (18)$$

This is shown by dotted lines in Fig. 2(a). Note that, in this regime, $\kappa^{-1}/\kappa^{-1}(s=0) > 1$ and it decreases to unity with increasing sE_R/E_F . Therefore, $\kappa^{-1}/\kappa^{-1}(s=0)$ should take a maximum value larger than unity in the intermediate regime of $E_F/E_R \sim 1$ as can be seen in Fig. 2(a), which is mainly caused by the bandgap in the longitudinal degree of freedom.

Since the tunneling rate between neighboring sites, which is related to the (inverse) effective mass $1/m^*$, is exponentially suppressed with the mass, the formation of molecules induced by the lattice can yield a drastic enhancement of m^* for $s \gg 1$ in the low density limit [Fig. 2(b)]. This enhancement makes m^* much larger compared to the ideal Fermi gases (see Fig. 2 in Ref. [29]) and to the BECs of repulsively interacting Bose gases (see Fig. 4 in Ref. [35]) with the same mass m . As E_F/E_R increases, the effective mass exhibits a maximum at $E_F/E_R \sim 1$ due to the bandgap in the transverse degree of freedom, then it decreases to the bare mass, $m^* = m$. Hydrodynamic theory can explain the behavior of m^* of unitary Fermi gases for small sE_R/E_F [29]:

$$\frac{m^*}{m} \simeq 1 + \frac{9}{32}(1+\beta)^{-2} \left(\frac{sE_R}{E_F} \right)^2 + O[(sE_R/E_F)^4]. \quad (19)$$

The numerical factor in the second term shows that the effect of the lattice is stronger for m^* than for κ^{-1} . It is worth comparing the results with the case of bosonic atoms, where m^* decreases monotonically with increasing density since the interaction broadens the condensate wave function and favors the tunneling [35].

In Fig. 3, we show the sound velocity of the unitary Fermi gases in a lattice,

$$c_s = \sqrt{\frac{\kappa^{-1}}{m^*}}, \quad (20)$$

calculated from the above results of κ^{-1} and m^* . It shows a significant reduction compared to the uniform system mainly due to the larger effective mass $m^*/m > 1$ except for the low density (more precisely, strong lattice) limit in which κ^{-1} and thus c_s show an abrupt reduction. Using Eqs. (18) and (19), we obtain the expression of the sound velocity in the weak lattice limit as

$$c_s^2 \simeq c_s^{(0)2} \left[1 - \frac{1}{4}(1+\beta)^{-2} \left(\frac{sE_R}{E_F} \right)^2 + O[(sE_R/E_F)^4] \right], \quad (21)$$

where $c_s^{(0)} \equiv [(2/3)(1+\beta)E_F/m]^{1/2}$ is the sound velocity for the uniform system.

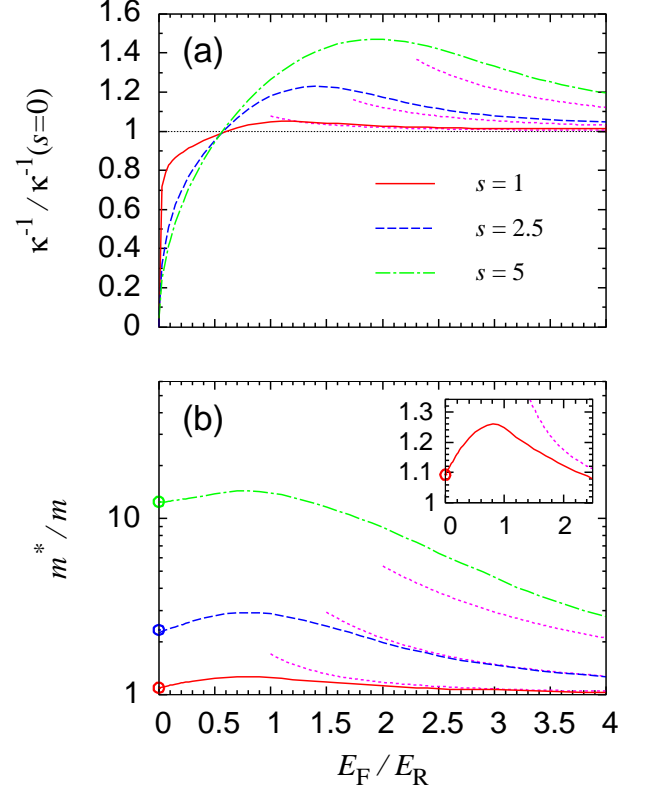


FIG. 2: (Color online) Incompressibility κ^{-1} , and effective mass m^* of unitary Fermi gases for $s = 1$ (red), 2.5 (blue), and 5 (green). Asymptotic expressions (18) and (19) obtained by the hydrodynamic theory are shown by the dotted lines. Open circles in panel (b) show m^* obtained in Ref. [30] by solving the Schrödinger equation for two-body problem. The $s = 1$ results for m^* are also shown in the inset in the linear scale. This figure is adapted from Ref. [29].

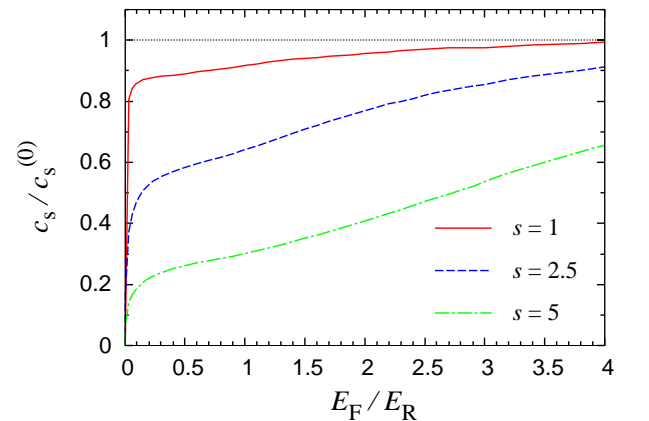


FIG. 3: (Color online) Sound velocity c_s of unitary Fermi gases in a lattice in units of the sound velocity $c_s^{(0)} = [(2/3)(1+\beta)E_F/m]^{1/2}$ for the uniform system. As in Fig. 2, red, blue and green lines correspond to $s = 1, 2.5$, and 5, respectively. This figure is taken from Ref. [29].

IV. STABILITY

A. Landau criterion

Stability of superfluid flow is one of the most fundamental issues of superfluidity, which was pioneered by Landau [36]. He predicted that there is a critical velocity v_c of the superflow above which the kinetic energy of the superfluid is large enough to be dissipated by creating excitations (see, e.g., Refs. [36–39]). This instability is called Landau or energetic instability and its critical velocity is Landau critical velocity.

The celebrated Landau criterion for energetic instability of uniform superfluids is given by [36–39]

$$v > v_c = \min \left(\frac{\epsilon(p)}{p} \right), \quad (22)$$

where v is the velocity of the superflow, $\epsilon(p)$ is the excitation spectrum in the static case ($v = 0$), and p is the magnitude of momentum \mathbf{p} of an excitation in the comoving frame of the fluid. Here, v_c is determined by the condition for which there starts to exist a momentum \mathbf{p} at which the excitation spectrum in the comoving frame of a perturber (it can be an obstacle moving in the fluid or a vessel in which the fluid flows) is zero or negative.

For superfluids of weakly interacting Bose gases, excitation spectrum is given by the Bogoliubov dispersion relation (e.g., Refs. [37–40]),

$$\epsilon(p) = \sqrt{\frac{p^2}{2m} \left(\frac{p^2}{2m} + 2gn \right)} = c_s p \sqrt{1 + \left(\frac{p}{2mc_s} \right)^2}, \quad (23)$$

with

$$c_s = \sqrt{\frac{gn}{m}} \quad (24)$$

being the sound velocity defined by Eq. (20) [note that $\mu = gn$ and thus $\kappa^{-1} = gn$, and $m^* = m$ for uniform BECs]. Thus, from Eqs. (22) and (24), it is easily seen that the Landau critical velocity is given by the sound velocity $v_c = c_s$ for $p = 0$. This means that, in superfluids of dilute Bose gases, the energetic instability is caused by excitations of long-wavelength phonons. It is also noted that BECs of non-interacting Bose gases cannot show the superfluidity in a sense that $v_c = 0$ and they cannot support a superflow.

In superfluid Fermi gases, there is another mechanism which can cause the energetic instability: fermionic pair-breaking excitations. In the mean-field BCS theory, quasiparticle spectrum of uniform superfluid Fermi gases is given by

$$\epsilon(p) = \sqrt{\left(\frac{p^2}{2m} - \mu \right)^2 + \Delta^2}. \quad (25)$$

Thus the Landau critical velocity due to the pair-breaking excitations is given by [41]

$$v_c = \sqrt{\frac{1}{m} \left(\sqrt{\mu^2 + \Delta^2} - \mu \right)}. \quad (26)$$

In the deep BCS regime, where $\mu \simeq E_F \gg \Delta$, we obtain $v_c \simeq \Delta/p_F$ with $p_F \equiv \hbar k_F$.

In the BCS-BEC crossover of superfluid Fermi gases, where the above two kinds of excitations exist, the Landau critical velocity is determined by either of them which gives smaller v_c . In the weakly interacting BCS regime ($1/k_F a_s \lesssim -1$), the pairing gap is exponentially small, $\Delta \sim E_F e^{-\pi/2k_F |a_s|}$, and v_c is set by the pair-breaking excitations. On the other hand, in the BEC regime ($1/k_F a_s \gtrsim 1$), where the system consists of weakly interacting bosonic molecules, creation of long-wavelength superfluid phonon excitations causes the energetic instability. In the unitary regime both mechanisms are suppressed and the critical velocity shows a maximum value [41–43].

B. Stability of superflow in lattice systems

1. Energetic instability and dynamical instability

For the onset of the energetic instability, energy dissipation is necessary in general; i.e., in closed systems, $v > v_c$ is not a sufficient condition for a breakdown of the superflow and the flow could still persist even at $v > v_c$. The energetic instability corresponds to the situation in which the system is located at a saddle point of the energy landscape (i.e., there is at least one direction in which the curvature of the energy landscape is negative).

In the presence of a periodic potential, another type of instability called dynamical (or modulational) instability can occur in addition to the energetic instability. The dynamical instability means that small perturbations on a stationary state grow exponentially in the process of (unitary) time evolution without dissipation. Similar to energetically unstable states, dynamically unstable states are also located at saddle points in the energy landscape, but a difference from energetically unstable states is that there are kinematically allowed excitation processes which satisfy the energy and (quasi)momentum conservation. This means that energetic instability is a necessary condition for dynamical instability (see, e.g., Refs. [44, 45] for bosons and [46] for fermions), and therefore, critical value of the (quasi)momentum for the dynamical instability should be always larger than (or equal to) that for the energetic instability. Due to such a reason, we shall focus on the energetic instability hereafter [47].

2. Determination of the critical velocity

If the energetic instability is caused by long-wavelength superfluid phonon excitations, the critical velocity can be determined by hydrodynamic analysis of the excitations [27, 38, 52–55] (this should not be confused with the LDA hydrodynamics discussed in Sec. II C). This analysis is valid provided that the wavelength of the excitations which trigger the instability is much larger than the typical length scale of the density variation, i.e., the lattice constant d .

We consider the continuity equation and the Euler equation for the coarse-grained density n and the coarse-grained (quasi)momentum P averaged over the length scale larger than the lattice constant,

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = \frac{\partial n}{\partial t} + \frac{\partial}{\partial z} \frac{\partial e}{\partial P} = 0, \quad (27)$$

$$\frac{\partial P}{\partial t} + \frac{\partial \mu}{\partial z} = \frac{\partial P}{\partial t} + \frac{\partial}{\partial z} \frac{\partial e}{\partial n} = 0, \quad (28)$$

where $e = e(n, P)$ is the energy density of the superfluid in the periodic potential for averaged density n and averaged (quasi)momentum P . Linearizing with respect to the perturbations of $n(z, t) = n_0 + \delta n(z, t)$ and $P(z, t) = P_0 + \delta P(z, t)$ with $\delta n(z, t)$ and $\delta P(z, t) \propto e^{iqz - i\omega t}$, we obtain the dispersion relation of the long-wavelength phonon,

$$\omega(q) = \frac{\partial^2 e}{\partial n \partial P} q + \sqrt{\frac{\partial^2 e}{\partial n^2} \frac{\partial^2 e}{\partial P^2}} |q|. \quad (29)$$

Here, $\hbar\omega$ and q are the energy and the wavenumber of the excitation. In the first term (so-called the Doppler term), $\partial_n \partial_P e \geq 0$. Thus the energetic instability occurs when $\omega(q)$ for $q = -|q|$ becomes zero or negative:

$$\frac{\partial^2 e}{\partial n \partial P} \geq \sqrt{\frac{\partial^2 e}{\partial n^2} \frac{\partial^2 e}{\partial P^2}}. \quad (30)$$

Using this condition, we determine the critical quasimomentum P_c at which the equality of Eq. (30) holds, and finally we obtain the critical velocity from

$$v_c = \frac{1}{n} \left(\frac{\partial e}{\partial P} \right)_{P_c}. \quad (31)$$

It is noted that, for calculating P_c and v_c using Eqs. (30) and (31), what we only need is the energy density of the stationary states as a function of n and P . This can be obtained by solving, e.g., GP or BdG equations for the periodic potential.

If the energetic instability of Fermi superfluids is caused by the pair-breaking excitations, the critical velocity can be determined by the quasiparticle energy spectrum ϵ_i obtained from the BdG equations. The energetic instability by the pair-breaking excitations occurs when some quasiparticle energy ϵ_i becomes zero or negative:

$$\epsilon_i \leq 0. \quad (32)$$

We obtain a critical velocity for the pair-breaking excitations from Eq. (31) evaluated at the critical quasimomentum determined by this condition.

3. Critical velocity of superfluid Bose gases and superfluid unitary Fermi gases in a lattice

First, we consider the situation where the LDA is valid, i.e., the lattice constant d is much larger than the healing length ξ . This condition corresponds to $gn/E_R \gg 1$ for superfluid Bose gases and $E_F/E_R \gg 1$ for superfluid Fermi gases at unitary (see discussion in Sec. II A). In the framework of the LDA hydrodynamics explained in Sec. II C, the system is considered to become energetically unstable if there exists some point z at which the local superfluid velocity $v(z)$ is equal to or larger than the local sound velocity $c_s(z)$. Suppose the external potential has a maximum at $z = z_0$ [i.e., $V(z_0) = V_0$ for our periodic potential], then at the same point the density is minimum, $c_s(z)$ is minimum, and $v(z)$ is maximum due to the current conservation, i.e., $j = n(z)v(z)$ being constant. This means that the superfluid becomes first unstable at $z = z_0$. Using Eq. (13), the condition for the occurrence of the instability can be written as $m[j_c/n_c(z_0)]^2 = \gamma\mu[n_c(z_0)] = \gamma\alpha n_c^\gamma(z_0)$, where $n_c(z)$ is the density profile calculated at the critical current [56]. By inserting this condition into Eq. (11), one can obtain the following implicit relation for the critical current [27]:

$$j_c^2 = \frac{\gamma}{m\alpha^{2/\gamma}} \left[\frac{2\mu_{j_c}}{2 + \gamma} \left(1 - \frac{V(z_0)}{\mu_{j_c}} \right) \right]^{\frac{2}{\gamma} + 1}. \quad (33)$$

It is worth noticing that this equation contains only z -independent quantities. It is also independent of the shape of the external potential: the only relevant parameter is its maximum value $V(z_0)$. Moreover, it can be applied to both Bose gases and unitary Fermi gases (see also Refs. [57–59] for bosons).

In Fig. 4, we plot the critical velocity in a lattice obtained from the hydrodynamic expression (33) for BECs [panel (a)] and unitary Fermi gases [panel (b)] by the thick solid red lines. In both cases, the critical velocity $v_c = j_c/n_0$ (n_0 is the average density) is normalized to the value of the sound velocity in the uniform gas, $c_s^{(0)}$, and is plotted as a function of $V_0/\mu_{j=0}$. The limit $V_0/\mu_{j=0} \rightarrow 0$ corresponds to the usual Landau criterion for a uniform superfluid flow in the presence of a small external perturbation, i.e., a critical velocity equal to the sound velocity of the gas. In this hydrodynamic scheme, as mentioned before, the critical velocity decreases when V_0 increases mainly because the density has a local depletion and the velocity has a corresponding local maximum, so that the Landau instability occurs earlier. When $V_0 = \mu_{j_c}$, the density exactly vanishes at $z = z_0$ and hence the critical velocity goes to zero.

Next, we discuss the critical velocity of the energetic instability beyond the LDA [27]. Here we use the energy

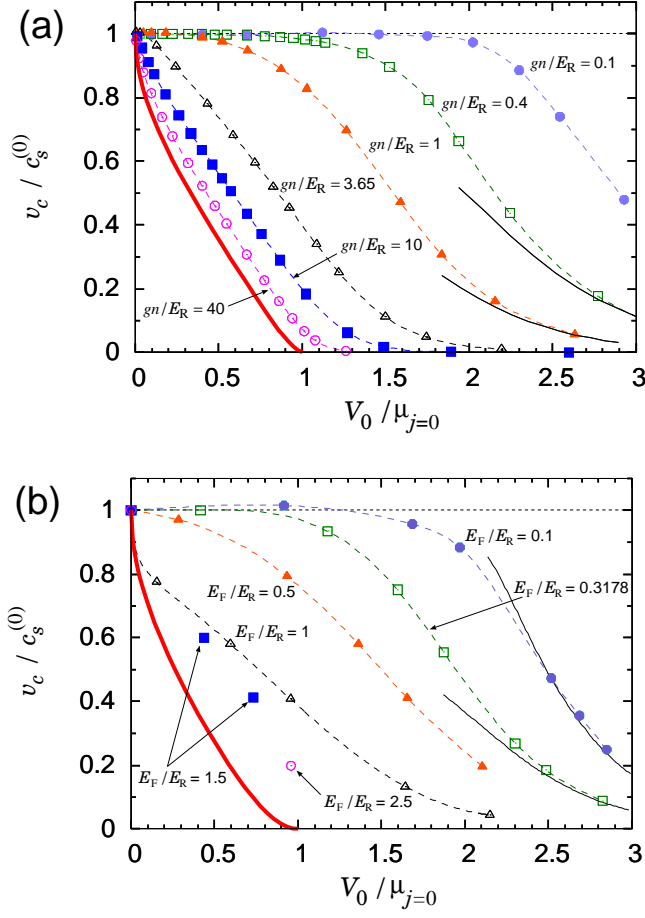


FIG. 4: (Color online) Critical velocity v_c for energetic instability of superfluids in a 1D periodic potential. Panel (a) is for superfluids of dilute Bose gases and panel (b) for superfluids of dilute Fermi gases at unitarity. The critical velocity is given in units of the sound velocity of a uniform gas, $c_s^{(0)}$, and is plotted as a function of the maximum of the external potential in units of the chemical potential $\mu_{j=0}$ of the superfluid at rest. Thick solid lines: prediction of the hydrodynamic theory within the LDA, calculated from Eq. (33). Symbols: results obtained from the numerical solutions of the GP equation [panel (a)] and the BdG equations [panel (b)]. The thinner black solid lines are the tight-binding prediction (35). Dashed lines are guides to the eye. This figure is adapted from Ref. [27].

density $e(n, P)$ of superfluids in a periodic potential calculated by the mean-field theory in the continuum model, i.e., the GP equation for bosons and the BdG equations for fermions. Based on the energy density, we determine the critical quasimomentum and the critical velocity from Eqs. (30) and (31).

For Bose superfluids, we plot v_c for various values of gn/E_R in Fig. 4(a). We can clearly see that these results approach the LDA prediction for $gn/E_R \gg 1$, as expected. We also note that v_c exhibits a plateau for $gn/E_R \lesssim 1$ (i.e., $\xi \gtrsim d$) and small V_0 . This can be understood as follows. If the healing length ξ is larger

than the lattice spacing d and V_0/gn is not too large, the energy associated with quantum pressure, which is proportional to $1/\xi^2$, acts against local deformations of the order parameter and the latter remains almost unaffected by the modulation of the external potential. This is the regime of the plateau in Fig. 4. In terms of Eq. (30), this regime occurs when the left-hand side is $\simeq P/m$ and the right-hand side is $\simeq c_s^{(0)}$, so that the critical quasimomentum obeys the relation $P_c/m = c_s^{(0)}$, which is the usual Landau criterion for a uniform superfluid in the presence of small perturbations. By increasing V_0 , this regime ends when $\mu \sim E_R$.

If we further increase V_0 , the chemical potential μ becomes larger than E_R , the density is forced to oscillate, and $v_c/c_s^{(0)}$ starts to decrease. The system eventually reaches a regime of weakly coupled superfluids separated by strong barriers, which is well described by the tight-binding approximation (also for $gn/E_R \gtrsim 1$, the system enters this regime when V_0 is sufficiently large). There, the energy density is given by the sinusoidal form with respect to P as

$$e(n, P) = e(n, 0) + \delta_J [1 - \cos(\pi P/P_{\text{edge}})] . \quad (34)$$

Here, P_{edge} is the quasimomentum at the edge of the first Brillouin zone, i.e., $P_{\text{edge}} = \hbar q_B$ for superfluids of bosonic atoms ($P_{\text{edge}} = \hbar q_B/2$ for those of fermionic atoms). The quantity $\delta_J = n P_{\text{edge}}^2 / \pi^2 m^*$ corresponds to the half width of the lowest Bloch band. Because of the sinusoidal shape of the energy density and large effective mass in the tight-binding limit, the critical quasimomentum is around $P_{\text{edge}}/2$. Thus we see that, from Eq. (31), the critical velocity is determined by the effective mass as

$$v_c \simeq \frac{1}{\pi} \frac{P_{\text{edge}}}{m^*} . \quad (35)$$

The values of v_c obtained from Eq. (35), with m^* extracted from the GP calculation of $e(n, P)$, are plotted in Fig. 4(a) by thinner black solid lines for $gn/E_R = 0.4$ and 1 in the region of $V_0/\mu_{j=0} \gtrsim 2$.

We calculate the critical velocity also by another method based on the complete linear stability analysis for the GP energy functional as in Refs. [44, 52, 60] (see also Refs. [38, 45]). We have checked that the results agree with those obtained from Eq. (30) based on the hydrodynamic analysis within 1% in the whole range of gn/E_R and V_0 considered in the present work. This confirms that the energetic instability in the periodic potential is triggered by long-wavelength excitations, since the excitation energy of the sound mode is the smallest in this limit.

For superfluid unitary Fermi gases, we plot v_c for various values of E_F/E_R in Fig. 4(b). We observe qualitatively similar results to those for bosons plotted in Fig. 4(a). For $E_F/E_R \lesssim 1$, the critical velocity v_c shows a plateau at small V_0 and it decreases from $\simeq c_s^{(0)}$ with increasing V_0 . For larger $E_F/E_R > 1$, v_c approaches the

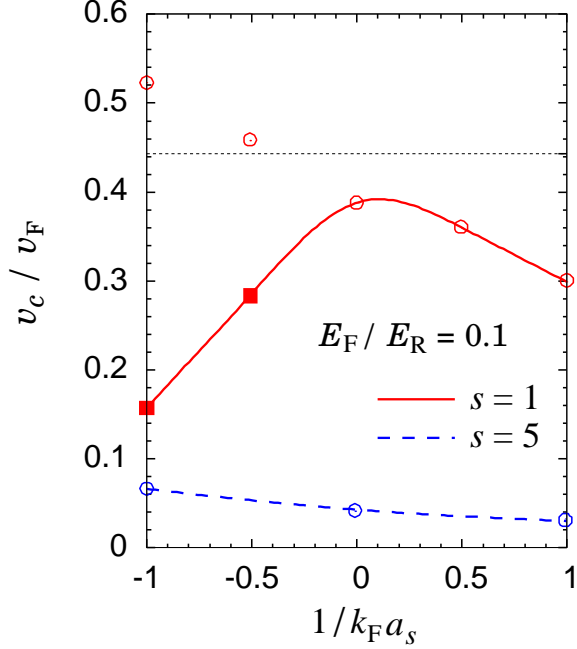


FIG. 5: (Color online) Critical velocity v_c of the energetic instability for $E_F/E_R = 0.1$ with $s = 1$ and 5 in the BCS-BEC crossover. Open circles and filled squares show the critical velocity due to long-wavelength phonons and fermionic pair-breaking excitations, respectively. The horizontal dotted line represents the value of the sound velocity $c_s^{(0)}$ of uniform system at unitarity, $c_s^{(0)}/v_F = (1 + \beta)^{1/2}/\sqrt{3} \simeq 0.443$. The red solid line is a guide to the eye.

LDA result (thick solid red line). In the regime of large V_0 such that V_0/μ is sufficiently large, v_c is well described by the tight-binding expression (35) plotted by thinner black solid lines. Here we use m^* calculated from BdG equations. We also note that the pair-breaking excitations are irrelevant to the energetic instability at unitarity except for very low densities such that $E_F/E_R \ll 1$ and small but nonzero V_0 . These excitations are important in the BCS side of the BCS-BEC crossover discussed in Sec. IV B 4.

Finally, we would like to point out that, in the LDA limit, while the quantities discussed in Sec. III approach the value in the uniform system, the critical velocity of the energetic instability is most strongly affected by the lattice. Figure 4 clearly shows that, as we increase the lattice height, the critical velocity decreases from that in the uniform system most rapidly in the LDA limit. The main reason of this opposite tendency is that the critical velocity is determined only around the potential maxima while the quantities discussed in Sec. III are determined by the contribution from whole region of the system.

4. Along the BCS-BEC crossover

Here we extend our discussion on the Landau critical velocity to the BCS-BEC crossover region [55]. As in the uniform systems, both the long-wavelength phonon excitations and pair-breaking excitations can be relevant to the energetic instability depending on the interaction parameter $1/k_F a_s$. However, there is an additional effect due to the lattice: when the lattice height is much larger than the Fermi energy, the periodic potential can make pairs of atoms to be strongly bound even in the BCS regime, and the pair-breaking excitations are suppressed [55].

In Fig. 5, we plot v_c for superfluid Fermi gases in the BCS-BEC crossover for different values of the lattice height with $s = 1$ and 5 . Here we set $E_F/E_R = 0.1$ as an example. The open circles show v_c of the energetic instability caused by the long-wavelength superfluid phonon excitations and the filled squares show that by the pair-breaking excitations. For moderate lattice strength of $s = 1$, the result of v_c is qualitatively the same as that of the uniform system. In the BCS side, smallest v_c is given by the pair-breaking excitations while, in the BEC side, v_c is set by the long-wavelength phonon excitations, and around unitarity v_c shows a maximum. For a larger value of $s = 5$, however, the result is very different from the uniform case. Due to the lattice-induced binding, pair breaking is suppressed so that it does not cause the energetic instability even in the BCS side (Note that there is no filled square for $s = 5$. This means that we do not have negative quasiparticle energy for any value of P in the whole Brillouin zone. This point will be discussed later.). Throughout the calculated region of $-1 \leq 1/k_F a_s \leq 1$, v_c is determined only by the long-wavelength phonon excitations and v_c decreases monotonically with increasing $1/k_F a_s$.

The effect of the lattice-induced molecular formation can be clearly seen in the quasiparticle energy spectrum ϵ_i and in the enhancement of the order parameter $|\Delta(z)|$. In Fig. 6, we show ϵ_i and $|\Delta(z)|$ in the case of $s = 5$, $E_F/E_R = 0.1$, and $1/k_F a_s = -1$. The spectrum for $P = 0$ shows a quadratic dependence of k_z with a positive curvature around $k_z = 0$ and there are no minima at $k_z \neq 0$. Even though the figure represents a case in the BCS regime, the structure of ϵ_i is consistent with the formation of bound pairs. We can also see that the spectrum never becomes negative for any values of P in the first Brillouin zone. In the inset of the same figure, we show the amplitude $|\Delta(z)|$ of the order parameter at $P = 0$. Here we see a large enhancement of $|\Delta(z)|$, near $z = 0$, compared to the uniform system, which shows the formation of bosonic bound molecules. We also note that the minimum value of $|\Delta(z)|$ at $z/d = \pm 1$ is smaller than, but still comparable to the value of $|\Delta|$ in the uniform case, suggesting that the system is indeed in the superfluid phase.

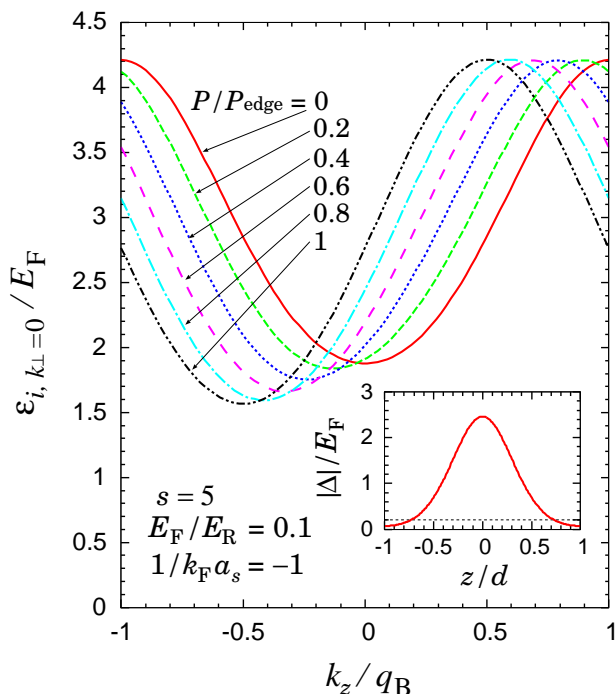


FIG. 6: (Color online) Lowest band of the quasiparticle energy spectrum ϵ_i for large lattice height with $s = 5$ and $E_F/E_R = 0.1$ (i.e., $E_F/V_0 = 0.02$) in the BCS regime at $1/k_F a_s = -1$. Here, we show the first radial branch with $k_{\perp}^2 \equiv k_x^2 + k_y^2 = 0$, which always gives the smallest values of ϵ_i in this case. The inset shows the amplitude $|\Delta(z)|$ of the order parameter at $P = 0$. The horizontal dotted line shows the amplitude of the order parameter for the uniform system at the same values of $1/k_F a_s = -1$. This figure is adapted from Ref. [55].

5. Experiments

In closing this section, we briefly summarize experimental studies on stability of the superfluid. Using cold atomic gases, stability of the superfluid and its critical velocity was first experimentally studied in Ref. [61] and further examined in Ref. [62]. In these experiments, they put a large (diameter $\gg \xi$) and strong (height $\gg \mu$) vibrating circular potential in a BEC. However, it has been concluded that what they observed in these works were not likely to be the energetic instability: dynamical nucleation of vortices by vibrating potential [62]. Recently, Ramanathan *et al.* performed a new experiment on stability of the superflow with a different setup [63]. They used a BEC flowing in a toroidal trap with a tunable weak link (width of the barrier along the flow direction is much larger than ξ). In this experiment, they obtained the critical velocity consistent with the energetic instability due to the vortex excitations [64]. For 2D Bose gases, their stability of the superfluidity and the critical velocity was also studied recently [65].

Regarding the superflow in optical lattices, its stabil-

ity was first experimentally studied in Ref. [66]. In this experiment, they used a cigar-shaped BEC which undergoes the center-of-mass oscillation in a harmonic trap in the presence of a weak 1D optical lattice, and measured the critical velocity. Although their original conclusion was that they have measured the Landau critical velocity for the energetic instability, further careful experimental [67] and theoretical [44, 60] follow-up studies have clarified that the instability observed in Ref. [66] was a dynamical instability. In this follow-up experiment by De Sarlo *et al.* [67], they employed an improved setup, i.e., a 1D optical lattice moving at constant and tunable velocities instead of an oscillating BEC in a static lattice. With this new setup, they succeeded in observing both energetic [67] and dynamical [67, 68] instabilities, and obtained a very good agreement with theoretical predictions [60, 67, 68]. It is worthwhile to stress that this understanding was finally obtained by a continuous effort over a few years by the same group.

Experimental study on stability of superfluid Fermi gases in optical lattices was carried out by Miller *et al.* [34]. Similar to the experiment of Ref. [67], they also used a 1D lattice moving at constant and tunable velocities. A different point is that, instead of imposing a periodic potential on the whole cloud, they produced a lattice potential only in the central region of the cloud. They measured the critical velocity of the energetic instability in the BCS-BEC crossover and found that it is largest around unitarity. They also did a systematic measurement of the critical velocity at unitarity for various lattice height. However, there is a significant discrepancy from a theoretical prediction [27] and further studies are needed.

V. ENERGY BAND STRUCTURE

In this section we discuss how the superfluidity can affect the energy band structures of ultracold atomic gases in periodic optical potentials. Starting with the noninteracting particles in a periodic potential, we obtain the well-known sinusoidal energy band structure. When the interparticle interaction is turned on in such a way that the superfluidity appears, we see a drastic change in energy band structure. For BECs, it has been pointed out that the interaction can change the Bloch band structure drastically, causing the appearance of a loop structure called “swallowtail” in the energy dispersion [69–72]. This is due to the competition between the external periodic potential and the nonlinear mean-field interaction: the former favors a sinusoidal band structure, while the latter tends to make the density smoother and the energy dispersion quadratic. When the nonlinearity wins, the effect of the external potential is screened and a swallowtail energy loop appears [73]. This nonlinear effect requires the existence of an order parameter and, consequently, the emergence of swallowtails can be viewed as a peculiar manifestation of superfluidity in periodic potentials.

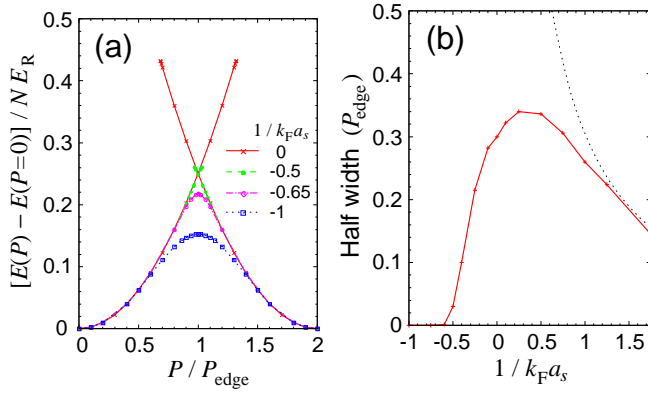


FIG. 7: (Color online) (a) Energy E per particle as a function of the quasimomentum P for various values of $1/k_F a_s$; (b) half-width of the swallowtails along the BCS-BEC crossover. These results are obtained for $s = 0.1$ and $E_F/E_R = 2.5$. The quasimomentum $P_{\text{edge}} = \hbar q_B/2$ fixes the edge of the first Brillouin zone. The dotted line in (b) is the half-width in a BEC obtained by solving the GP equation; it vanishes at $1/k_F a_s \simeq 10.6$. This figure is taken from Ref. [74].

Qualitatively, one can argue in a similar way on the existence of the swallowtail energy band structure in Fermi superfluids in optical lattices; the competition between the external periodic potential energy and the nonlinear interaction energy ($g|\sum_i u_i(\mathbf{r})v_i^*(\mathbf{r})|^2$) determines the energy band structure. However, the interaction energy in the Fermi gas is more involved and the unified view along the crossover from the BCS to the BEC states are non-trivial and interesting problems. To answer for the questions of 1) whether or not swallowtails exist in Fermi superfluids and 2) whether there exist unique features which make them different from those in bosons, we solve the BdG equations [see Eq. (9)] of a two-component unpolarized dilute Fermi gas subject to the one-dimensional (1D) optical lattice [74].

A. Swallowtail energy spectrum

The energy per particle in the lowest Bloch band as a function of the quasimomentum P for various values of $1/k_F a_s$ is computed [75]. The results in Fig. 7(a) show that the swallowtails appear above a critical value of $1/k_F a_s$ where the interaction energy is strong enough to dominate the lattice potential. In Fig. 7(b), the half-width of the swallowtails from the BCS to the BEC side is shown. It reaches a maximum near unitarity ($1/k_F a_s = 0$). In the far BCS and BEC limits, the width vanishes because the system is very weakly interacting and the band structure tends to be sinusoidal. When approaching unitarity from either side, the interaction energy increases and can dominate over the periodic potential, which means that the system behaves more like a translationally invariant superfluid and the band structure follows a quadratic dispersion terminating at a maxi-

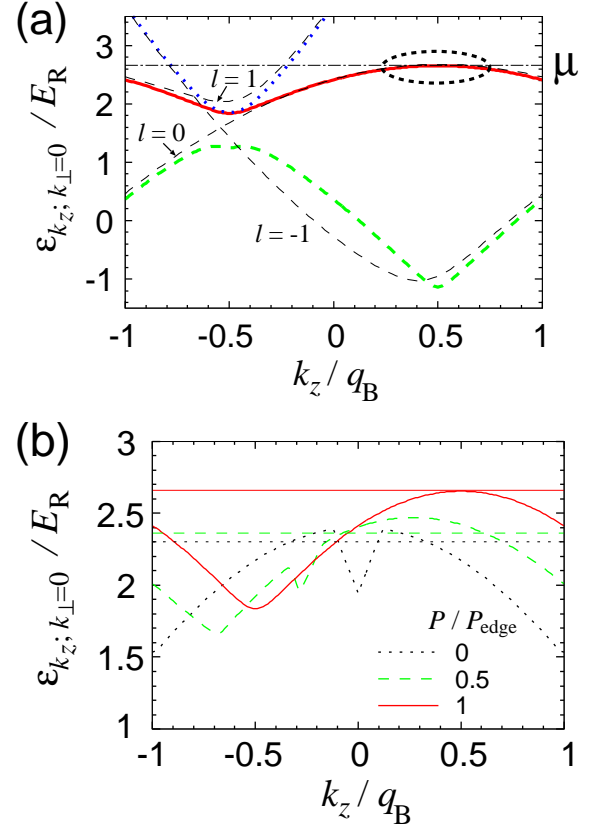


FIG. 8: (Color online) (a) Lowest three Bloch bands of the quasiparticle energy spectrum at $k_\perp = 0$ for $P = P_{\text{edge}}$ and $1/k_F a_s = -0.62$. Thin black dashed lines labeled by l 's show the approximate energy bands obtained from Eq. (36) using $\mu \simeq 2.66E_R$ and $|\Delta| \simeq |\Delta(0)| \simeq 0.54E_R$. (b) Bloch band of the quasiparticle energy spectrum around the chemical potential for $P/P_{\text{edge}} = 0$ (black dotted), 0.5 (green dashed), and 1 (red solid), respectively, at $k_\perp = 0$ and $1/k_F a_s = -0.62$ in the case of $s = 0.1$ and $E_F/E_R = 2.5$. The each horizontal line denotes the value of the chemical potential for the corresponding value of P . This figure is adapted from Ref. [74].

imum P larger than P_{edge} .

The emergence of swallowtails in the BCS side and for $E_F/E_R \gtrsim 1$ is associated with peculiar structures of the quasiparticle energy spectrum around the chemical potential. In the presence of a superflow moving in the z direction with wavevector Q ($\equiv P/\hbar$), the quasiparticle energies are given by the eigenvalues in Eq. (9). Since the potential is shallow ($s \ll 1$), some qualitative results can be obtained even ignoring $V_{\text{ext}}(z)$ except for its periodicity. With this assumption we obtain

$$\epsilon_{\mathbf{k}} \approx \frac{(k_z + 2q_B l)Q}{m} + \sqrt{\left[\frac{k_\perp^2 + (k_z + 2q_B l)^2 + Q^2}{2m} - \mu \right]^2 + |\Delta|^2}, \quad (36)$$

with l being integers for the band index. If $Q = 0$, the $l = 0$ band has the energy spectrum $\sqrt{[(k_\perp^2 + k_z^2)/2m - \mu]^2 + |\Delta|^2}$ which has a local maxi-

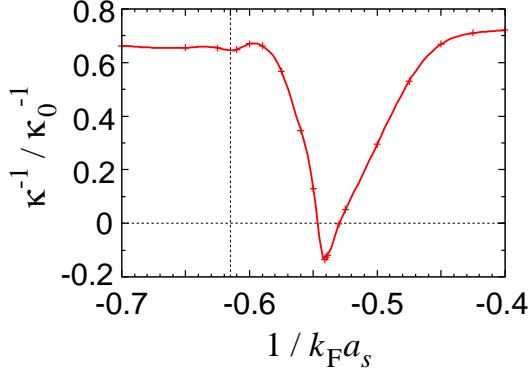


FIG. 9: Incompressibility κ^{-1} at $P = P_{\text{edge}}$ around the critical value of $1/k_F a_s \approx -0.62$ where the swallowtail starts to appear. The quantity κ_0^{-1} is the incompressibility of the homogeneous free Fermi gas of the same average density. In both panels, we have used the values $s = 0.1$ and $E_F/E_R = 2.5$. This figure is adapted from Ref. [74].

mum at $k_z = k_\perp = 0$. When $Q = P_{\text{edge}}/\hbar$, the spectrum is tilted and the local maximum moves to $k_z \simeq q_B/2$ provided $|\Delta| \ll E_F$ (and $E_F/E_R \gtrsim 1$). In the absence of the swallowtail, the full BdG calculation indeed gives a local maximum at $k_z = q_B/2$ and the quasiparticle spectrum is symmetric about this point, which reflects that the current is zero. As E_F/E_R increases, the band becomes flatter as a function of k_z and narrower in energy.

In Fig. 8(a), we show the quasiparticle energy spectrum at $k_\perp = 0$ for $P = P_{\text{edge}}$. When the swallowtail is on the edge of appearing, the top of the narrow band just touches the chemical potential μ [see the dotted ellipse in Fig. 8(a)]. Suppose $1/k_F a_s$ is slightly larger than the critical value, so that the top of the band is slightly above μ . In this situation a small change of the quasimomentum P causes a change of μ . In fact, when P is increased from $P = P_{\text{edge}}$ to larger values, the band is tilted and the top of the band moves upwards; the chemical potential μ should also increase to compensate for the loss of states available as shown in Fig. 8(b). This implies $\partial\mu/\partial P > 0$. On the other hand, since the system is periodic, the existence of a branch of stationary states with $\partial\mu/\partial P > 0$ at $P = P_{\text{edge}}$ implies the existence of another symmetric branch with $\partial\mu/\partial P < 0$ at the same point, thus suggesting the occurrence of a swallowtail structure.

B. Incompressibility

A direct consequence of the existence of a narrow band in the quasiparticle spectrum near the chemical potential is a strong reduction of the incompressibility $\kappa^{-1} = n\partial\mu(n)/\partial n$ close to the critical value of $1/k_F a_s$ in the region where swallowtails start to appear in the BCS side (see Fig. 9). The dip of κ^{-1} occurs in the situation where the top of the narrow band is just above μ for $P = P_{\text{edge}}$ ($1/k_F a_s$ is slightly above the critical value).

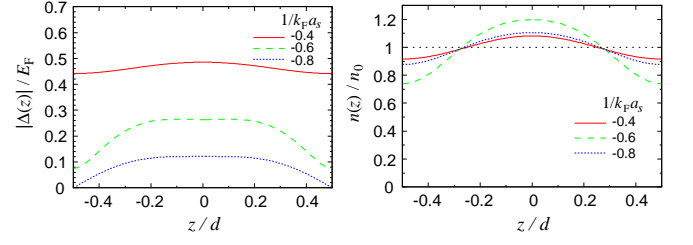


FIG. 10: Profiles of the pairing field $|\Delta(z)|$ and the density $n(z)$ at $1/k_F a_s = -0.8$ (blue dotted), -0.6 (green dashed), and -0.4 (red solid) for $P = P_{\text{edge}}$ in the case of $s = 0.1$ and $E_F/E_R = 2.5$. The swallowtail starts to appear at a critical value of $1/k_F a_s \approx -0.62$. This figure is taken from Ref. [74].

An increase of the density n has little effect on μ in this case, because the density of states is large in this range of energy and the new particles can easily adjust themselves near the top of the band by a small increase of μ . This implies that $\partial\mu(n)/\partial n$ is small and the incompressibility has a pronounced dip [76]. It is worth noting that in the BEC side the appearance of the swallowtail is not associated with any significant change of incompressibility. In fact, for the Bose gas of the average density n_{b0} of bosons, the exact solution of the GP equation gives $\kappa^{-1} = n_{b0}U_0$ near the critical conditions for the occurrence of swallowtails, being a smooth and monotonic function of the interaction strength.

C. Profiles of density and pairing fields

Both the pairing field and the density exhibit interesting features in the range of parameters where the swallowtails appear. This is particularly evident at the Brillouin zone boundary, $P = P_{\text{edge}}$. The full profiles of $|\Delta(z)|$ and $n(z)$ along the lattice vector (z -direction) are shown in Fig. 10. In general, $n(z)$ and $|\Delta(z)|$ take maximum (minimum) values where the external potential takes its minimum (maximum) values. By increasing the interaction parameter $1/k_F a_s$, we find that the order parameter $|\Delta|$ at the maximum ($z = \pm d/2$) of the lattice potential exhibits a transition from zero to nonzero values at the critical value of $1/k_F a_s$ at which the swallowtail appears. Note that here we plot the absolute value of Δ ; the order parameter Δ behaves smoothly and changes sign across zero.

In Fig. 11, we show the magnitude of the pairing field $|\Delta(z)|$ and the density $n(z)$ calculated at the minimum ($z = 0$) and at the maximum ($z = \pm d/2$) of the lattice potential. The figure shows that $|\Delta(d/2)|$ remains zero in the BCS regime until the swallowtail appears at $1/k_F a_s \approx -0.62$. Then it increases abruptly to values comparable to $|\Delta(0)|$, which means that the pairing field becomes almost uniform at $P = P_{\text{edge}}$ in the presence of swallowtails. As regards the density, we find that the amplitude of the density variation, $n(0) - n(d/2)$, exhibits a pronounced maximum near the critical value of $1/k_F a_s$.

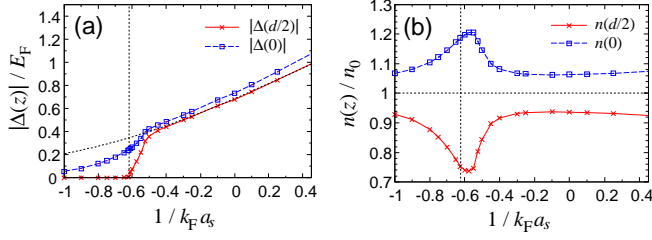


FIG. 11: (Color online) Profiles of (a) the pairing field $|\Delta(z)|$ and (b) the density $n(z)$ along the change of $1/k_F a_s$ for $P = P_{\text{edge}}$ in the case of $s = 0.1$ and $E_F/E_R = 2.5$. The values of $|\Delta(z)|$ and $n(z)$ at the minimum ($z = 0$, blue \square) and at the maximum ($z = \pm d/2$, red \times) of the lattice potential are shown. The vertical dotted lines show the critical value of $1/k_F a_s$ above which the swallowtail exists. The dotted curve in (a) shows $|\Delta|$ in the uniform system. This figure is taken from Ref. [74].

In contrast, in the BEC side, the order parameter and the density are smooth monotonic functions of the interaction strength even in the region where the swallowtail appears. At $P = P_{\text{edge}}$, the solution of the GP equation for bosonic dimers gives the densities $n_b(0) = n_{b0}(1 + V_0/2n_{b0}U_0)$ and $n_b(d/2) = n_{b0}(1 - V_0/2n_{b0}U_0)$ with $V_0/2n_{b0}U_0 = (3\pi/4)(sE_R/E_F)(1/k_F a_s)$, where $U_0 = 4\pi\hbar^2 a_b/m_b$ [69, 77, 78]. Near the critical value of $1/k_F a_s$, unlike the BCS side, the nonuniformity just decreases all the way even after the swallowtail appears. The local density at $z = d/2$ is zero until the swallowtail appears in the BEC side while it is nonzero in the BCS side irrespective of the existence of the swallowtail. The qualitative behavior of $|\Delta(z)|$ around the critical point of $1/k_F a_s$ is similar to that of $n_b(z)$ because $n_b(\mathbf{r}) = (m^2 a_s/8\pi)|\Delta(\mathbf{r})|^2$ [79] in the BEC limit.

VI. QUANTUM PHASES OF COLD ATOMIC GASES IN OPTICAL LATTICES

So far, we have discussed some superfluid features of the cold atomic gases in an optical lattice mainly focusing on our works. In this section we discuss some important topics regarding various quantum phases that the cold atomic gases on optical lattices can show or simulate. In this section, strong enough periodic potentials to ignore the next-nearest hopping and small average numbers of particles per lattice site will be assumed so that the system is described by the Hubbard lattice model.

A. Quantum phase transition from superfluid to Mott insulator

At a temperature of absolute zero, cold atomic gases in optical lattices can undergo a quantum phase transition from superfluid to Mott insulator phases as the interaction strength between atoms is tuned from weakly

to strongly interacting regime [80]. The quantum phase transition of an interacting boson gas in a periodic lattice potential can be captured by the following Bose-Hubbard Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} (\hat{b}_i^\dagger \hat{b}_j + \text{h.c.}) + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1),$$

where $\hat{b}_i, \hat{b}_i^\dagger$ are annihilation and creation operators of a bosonic atom on the i -th lattice site and $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$ is the atomic number operator on the i -th site. The first term is the kinetic energy term whose strength is characterized by the hopping matrix element J between adjacent sites $\langle i, j \rangle$ and $U(> 0)$ in the second term is the strength of a short-range repulsive interactions between bosonic atoms.

In the limit where the kinetic energy dominates, the many-body ground state of N atoms and M lattice sites is a superposition of delocalized Bloch states with lowest energy (quasimomentum $\mathbf{k} = 0$):

$$|\Psi_{\text{SF}}\rangle_{U/J=0} \propto (\hat{b}_{\mathbf{k}=0}^\dagger)^N |0\rangle \propto \left(\sum_{i=1}^M \hat{b}_i^\dagger \right)^N |0\rangle, \quad (37)$$

where $|0\rangle$ is the empty lattice. This state has perfect phase correlation between atoms on different sites while the number of atoms on each site is not fixed. This superfluid phase has gapless phonon excitations.

In the limit where the interactions dominates (so-called, “atomic limit”), the fluctuations in the local atom number become energetically unfavorable and the ground state is made up of localized atomic wavefunctions with a fixed number of atoms per lattice site. The many-body ground state with a commensurate filling of n atoms per lattice site is given by

$$|\Psi_{\text{MI}}\rangle_{J/U=0} \propto \prod_{i=1}^M (\hat{b}_i^\dagger)^n |0\rangle. \quad (38)$$

There is no phase correlation between different sites because the energy is independent of the phases of wavefunctions on each site. This Mott insulator state, unlike the superfluid state, cannot be described by a macroscopic wavefunction. The lowest excited state can be obtained by moving one atom from one site to another, which gives the energy gap of $\Delta = (U/2)[(n+1)^2 + (n-1)^2 - 2n^2] = U$.

As the ratio of the interaction term to the tunneling term increases (U/J can be controlled by changing the depth V_0 of the optical lattice even without using the Feshbach resonances), the system will undergo a quantum phase transition from the superfluid state to the Mott insulator state accompanying the opening of the energy gap in the excitation spectrum. Greiner *et al.* realized experimentally the quantum phase transition from a Bose-Einstein condensate of ^{87}Rb atoms with weak repulsive interactions to a Mott insulator in a three-dimensional

optical lattice potential [16]. Notably, they could induce reversible changes between these two ground states by changing the strength V_0 of the optical lattice. The superfluid to Mott insulator transitions were also achieved in one- and two-dimensional cold atomic Bose gases [81].

Mott insulator phases of atomic Fermi gases with *repulsive* interactions on the three-dimensional optical lattice have been realized and the entrance into the Mott insulating states was observed by verifying vanishing compressibility as well as by measuring the suppression of doubly occupied sites [82, 83].

B. Quantum phase transition from paramagnet to antiferromagnet

Sachdev *et al.* showed that the one-dimensional Mott insulator of spinless bosons in a tilted optical lattice can be mapped onto a quantum Ising chain [84]. The Bose-Hubbard Hamiltonian for a tilted optical lattice takes the form:

$$H = -J \sum_i (\hat{b}_i \hat{b}_{i+1}^\dagger + \hat{b}_i^\dagger \hat{b}_{i+1}) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - E \sum_i i \hat{n}_i,$$

where E is the lattice potential gradient per lattice spacing. For a tilt near $E = U$, the energy cost moving an atom to its neighbor (from site i to site $i + 1$) is zero. If we start with a Mott insulator with a single atom per site, an atom can resonantly tunnel into the neighboring site to produce a dipole state (at the link) consisting of a quasihole-quasiparticle pair on nearest neighbor sites. Only one dipole can be created per link and neighboring links cannot support dipoles together. This nearest-neighbor constraint is the source of the effective dipole-dipole interaction which results in a density wave ordering. Defining a dipole creation operator $\hat{d}_i^\dagger = \hat{b}_i \hat{b}_{i+1}^\dagger / \sqrt{2}$, the Bose-Hubbard Hamiltonian is mapped onto the dipole Hamiltonian:

$$H = -\sqrt{2}J \sum_i (\hat{d}_i^\dagger + \hat{d}_i) + (U - E) \sum_i \hat{d}_i^\dagger \hat{d}_i$$

with the constraint $\hat{d}_i^\dagger \hat{d}_i \leq 1$ and $\hat{d}_i^\dagger \hat{d}_i \hat{d}_{i+1}^\dagger \hat{d}_{i+1} = 0$. If the dipole present/absent link is identified with a pseudospin up/down, $\hat{S}_i^z = \hat{d}_i^\dagger \hat{d}_i - 1/2$, the pseudospin-1/2 Hamiltonian takes the form of a quantum Ising chain:

$$\begin{aligned} H &= J_S \sum_i \hat{S}_i^z \hat{S}_{i+1}^z - 2\sqrt{2}J \sum_i \hat{S}_i^x + (J_S - D) \sum_i \hat{S}_i^z \\ &= J_S \sum_i (\hat{S}_i^z \hat{S}_{i+1}^z - h_x \hat{S}_i^x + h_z \hat{S}_i^z), \end{aligned}$$

where $D = E - U$ and $\sum_i J_S (\hat{S}_i^z + 1/2) (\hat{S}_{i+1}^z + 1/2)$ with $J_S \rightarrow \infty$ is added to the Hamiltonian for implementing the constraint $\hat{d}_i^\dagger \hat{d}_i \hat{d}_{i+1}^\dagger \hat{d}_{i+1} = 0$. The dimensionless transverse and longitudinal fields are defined as $h_x = 2^{3/2}J/J_S$ and $h_z = 1 - D/J_S$.

Thereby a quantum phase transition from a paramagnetic phase ($D < 0$) to an antiferromagnetic phase ($D > 0$) can be studied by changing the lattice potential gradient E between adjacent sites. Using ^{87}Rb atoms quantum simulation of antiferromagnetic spin chains in an optical lattice was made and the phase transition to the antiferromagnetic phase from the paramagnetic phase was observed [85].

VII. SUMMARY AND OUTLOOK

In this article, we have focused on some superfluid properties of cold atomic gases in an optical lattice periodic along one spatial direction; basic macroscopic and static properties, stability of the superflow, and peculiar energy band structure. As a complement, some phases other than the superfluid phase have been discussed in the last section. Considering the rapid growth and interdisciplinary nature of the researches on cold atomic gases in optical lattices, it is practically impossible to cover all perspectives. We conclude by mentioning a few of them.

High controllability of the lattice potential opens many possibilities: Optical disordered lattice can be constructed to study the problem of Anderson localization of matter waves and the resulting phases [86]. The time modulation of the optical lattice was shown to tune the magnitude and the sign of the tunnel coupling in Hubbard model, which enables us to study various phases [87]. Cold atomic gases in optical lattices might uncover many exotic phases which are still in debate or even lack solid state analogs.

Due to their long characteristic time scales and large characteristic length scales, cold atomic gases are good playground for the experimental observation and control of their dynamics. Particularly, a sudden quench can be realized experimentally. The non-equilibrium dynamics after sudden quenches can be studied with high precision in experiments to discriminate between candidate theories. The long-standing problems such as thermalization, its connection with non-integrability and/or quantum chaos are actively sought-after topics in this direction [88].

The charge neutrality of atoms seems to limit the use of this system as a quantum simulator. However, internal states of an atom, together with atom-laser interaction, can be exploited for the atom to gain geometric Berry's phases, which amounts to generating artificial gauge fields interacting with these charge-neutral particles [89]. In this way, interactions with electromagnetic fields, spin-orbit coupling, and even non-abelian gauge fields can be emulated to open the door to study the physics of quantum Hall effects, topological superconductors/insulators, and high energy physics using cold atoms in the future.

Cold atomic gases provide a promising platform for controlling dissipation in addition to engineering Hamiltonian, e.g., by controlling the coupling with a subsystem

acting as a reservoir and by using external fields which induce losses of trapped atoms, etc. This possibility will allow us to use the cold atomic gases as quantum simulators of open systems. In addition, the controlled dissipation will offer an opportunity to study quantum dynamics driven by dissipation and its steady states, non-equilibrium phase transitions among the steady states determined by the competition between the coherent and dissipative dynamics. Furthermore, the controlling dissipation will pave a way to design Liovillian and dissipation-driven state preparation (e.g., Ref. [90]).

Last but not least, cold atomic gases in optical lattices are also useful for the precision measurements; “Optical lattice clock” consists of millions of atomic clocks working in parallel trapped in an optical lattice. The large number of the simultaneously-interrogated atoms greatly improves the stability of the clock and the state-of-the-art optical lattice clocks outperform the primary frequency standard of Cs clocks (Ref. [17] and references therein). Bloch oscillations of cold atomic gases in optical lattices offer a promising way of measuring forces at a spatial resolution of few micrometers (e.g., Ref. [91]). The pre-

cision measurement devices/techniques will enable high precision tests of time and space variation of fundamental constants, the weak equivalence principle, and Newton’s gravitational law in short distances.

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